

Physics 1A Discussion (Week 5)

Problem 1 (Particle Trajectories Near a Black Hole)

In general relativity, a black hole is described by the *Schwarzschild metric*

$$ds^2 = \left(1 - \frac{2GM}{r}\right) dt^2 - \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2)$$

where G is Newton's constant, M is the mass of the black hole, and (r, θ, ϕ) are spherical coordinates.

(a) Suppose you are pushed inside the black hole, meaning $r < 2GM$. You are allowed to flail around, throw stuff into the black hole, or do anything else you think might help you escape. The only condition is that, because you are massive, your displacement must always be timelike. Show that

$$\left| \frac{dr}{d\tau} \right| \geq \left(\frac{2GM}{r} - 1 \right)^{1/2} > 0$$

where τ is your proper time. In other words, assuming that $dr/d\tau$ is initially negative, you cannot turn around and escape the black hole.

Since your displacement must be always timelike

$$ds^2 = d\tau^2 > 0. \quad (1)$$

Then

$$d\tau^2 = - \left(\frac{2GM}{r} - 1 \right) dt^2 + \left(\frac{2GM}{r} - 1 \right)^{-1} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2) > 0 \quad (2)$$

$$\Rightarrow 1 = - \left(\frac{2GM}{r} - 1 \right) \left(\frac{dt}{d\tau} \right)^2 + \left(\frac{2GM}{r} - 1 \right)^{-1} \left(\frac{dr}{d\tau} \right)^2 - r^2 \left[\left(\frac{d\theta}{d\tau} \right)^2 + \sin^2 \theta \left(\frac{d\phi}{d\tau} \right)^2 \right]. \quad (3)$$

Since we have

$$r < 2GM \Rightarrow \frac{2GM}{r} > 1, \quad (4)$$

the first and the third term turns out to be negative. Thus we get

$$\left(\frac{2GM}{r} - 1 \right)^{-1} \left(\frac{dr}{d\tau} \right)^2 \geq 1 \quad (5)$$

$$\left| \frac{dr}{d\tau} \right| \geq \left(\frac{2GM}{r} - 1 \right)^{1/2} > 0 \quad (6)$$

(b) Now suppose you are (more comfortably) outside the black hole and at rest with respect to it. Your mass is m . Find your four-momentum p^μ in terms of m , G , and m . (Hint: the relationship $p^2 = m^2$ still works in general relativity, but now you need to use the new metric to lower the index of p^μ .)

Since you are outside of the blackhole and at rest respect to it, the spatial components of your four momentum must be zero $\mathbf{p} = 0$

$$p^\mu = \begin{pmatrix} p^0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \quad (7)$$

Then, the index can be lowered by

$$\begin{aligned} p_\mu &= g_{\mu\nu} p^\nu \\ &= \begin{pmatrix} 1 - \frac{2GM}{r} & 0 & 0 & 0 \\ 0 & -\left(1 - \frac{2GM}{r}\right)^{-1} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2 \sin^2 \theta \end{pmatrix} \begin{pmatrix} p^0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\ &= \left(\left(1 - \frac{2GM}{r}\right) p^0 \quad 0 \quad 0 \quad 0 \right) \end{aligned} \quad (8)$$

These four momenta must satisfy

$$\begin{aligned} m^2 &= p^\mu p_\mu \\ &= \left(1 - \frac{2GM}{r}\right) (p^0)^2. \end{aligned} \quad (9)$$

Then we get

$$\begin{aligned} p^0 &= m \left(1 - \frac{2GM}{r}\right)^{-1/2} \\ \Rightarrow p^\mu &= \begin{pmatrix} m \left(1 - \frac{2GM}{r}\right)^{-1/2} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad p_\mu = \left(m \left(1 - \frac{2GM}{r}\right)^{1/2} \quad 0 \quad 0 \quad 0 \right) \end{aligned} \quad (10)$$

(c) This time, you (mass m , four-momentum p^μ) remain at rest a distance r_1 from the center of the black hole, and you send a photon to your friend (also mass m , four-momentum q^μ) at rest a distance r_2 from the center of the black hole.

The four-momentum of the photon k^μ is now pretty complicated. Instead of investigating it directly, we now *define* the angular frequency ω of a photon, as measured by an observer with mass m and four-momentum p^μ , by the equation $k_\mu p^\mu = m\omega$. Defining it as a dot product ensures that our answer is consistent across all possible reference frames.

Because the Schwarzschild metric is static, it turns out that the 0th component of the photon four-momentum k^0 is conserved. Using this information, plus your result from part (b), solve for the ratio of the wavelengths observed by you and your friend, λ_2/λ_1 .

$k_\mu p^\mu = p_\mu k^\mu$ since dot products are symmetric. Then we get

$$\begin{aligned} m\omega_1 &\equiv k_\mu p^\mu = p_\mu k^\mu \\ &= \begin{pmatrix} m \left(1 - \frac{2GM}{r_1}\right)^{1/2} & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} k^0 \\ k^1 \\ k^2 \\ k^3 \end{pmatrix} \\ &= mk^0 \left(1 - \frac{2GM}{r_1}\right)^{1/2} \end{aligned} \quad (11)$$

In the same manner, we get

$$\begin{aligned} m\omega_2 &\equiv k_\mu q^\mu = q_\mu k^\mu \\ &= \begin{pmatrix} m \left(1 - \frac{2GM}{r_2}\right)^{1/2} & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} k^0 \\ k^1 \\ k^2 \\ k^3 \end{pmatrix} \\ &= mk^0 \left(1 - \frac{2GM}{r_2}\right)^{1/2}. \end{aligned} \quad (12)$$

Therefore,

$$\begin{aligned} \frac{\omega_2}{\omega_1} &= \frac{\left(1 - \frac{2GM}{r_2}\right)^{1/2}}{\left(1 - \frac{2GM}{r_1}\right)^{1/2}} \\ \Rightarrow \frac{\lambda_2}{\lambda_1} &= \frac{\omega_1}{\omega_2} = \left(\frac{1 - \frac{2GM}{r_1}}{1 - \frac{2GM}{r_2}} \right)^{1/2}. \end{aligned} \quad (13)$$

If $r_2 > r_1$, then $\lambda_2 > \lambda_1$. Your friend receives the photon as gravitationally redshifted.

Problem 2 (The Friedmann-Lemaître-Robertson-Walker Metric)

The Friedmann-Lemaître-Robertson-Walker (FLRW) metric, which describes the Universe on large scales, is given by

$$ds^2 = dt^2 - a(t)^2 \left[\frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \right]$$

Here, $a(t)$ is called the *scale factor*, a time-dependent parameter that describes how distances in the Universe expand. k is the *curvature parameter*, which defines the spatial curvature of the Universe¹. $k = 0$ for a flat universe, and we have measured $k = 0$ to reasonable precision for our Universe.

(a) Due to cosmic expansion, a galaxy is located at a comoving coordinate r . Express the proper distance $d_{\text{proper}}(t)$ to the galaxy in terms of the scale factor $a(t)$.

The proper distance to a galaxy at comoving coordinate r is given by integrating over the FLRW metric

$$d_{\text{proper}}(t) = a(t) r. \quad (14)$$

This means that while the comoving coordinate r remains fixed, the proper distance changes as the universe expands.

(b) Derive the proper velocity $v_{\text{proper}}(t)$ and show its relation to the proper distance. You can use the Hubble parameter $H(t) = \dot{a}/a$.

Differentiating the proper distance with respect to time

$$\begin{aligned} v_{\text{proper}}(t) &= \frac{d}{dt} (a(t) r) \\ &= \dot{a}(t) r \\ &= \frac{\dot{a}(t)}{a(t)} (a(t) r) \\ &\equiv H(t) d_{\text{proper}}(t). \end{aligned} \quad (15)$$

This is Hubble's law.

(c) For a light ray in the FLRW metric, the null condition $ds^2 = 0$ must hold. In a flat universe ($k = 0$), consider a light ray emitted from $r = 0$ at time t_{emit} and observed at time t_{obs} . Find the trajectory $r(t)$.

For a light ray, the null condition $ds^2 = 0$ gives

$$0 = -dt^2 + a^2(t) dr^2. \quad (16)$$

Since light moves along $dr/dt = 1/a(t)$, integrating from t_{emit} to t_{obs} , we obtain

$$\int_0^r dr' = \int_{t_{\text{emit}}}^{t_{\text{obs}}} \frac{dt'}{a(t')}. \quad (17)$$

Thus, the comoving distance the light has traveled is

$$r = \int_{t_{\text{emit}}}^{t_{\text{obs}}} \frac{dt'}{a(t')}. \quad (18)$$

(d) Compute the lookback distance, *i.e.*, the proper distance traveled by the light.

The proper distance to the emitting source at time t_{obs} is

$$d_{\text{proper}}(t_{\text{obs}}) = a(t_{\text{obs}}) r. \quad (19)$$

Substituting r , we get

$$d_{\text{proper}}(t_{\text{obs}}) = a(t_{\text{obs}}) \int_{t_{\text{emit}}}^{t_{\text{obs}}} \frac{dt'}{a(t')}. \quad (20)$$

¹This is different from the scalar curvature, or Ricci scalar R , which defines the spacetime curvature

This integral represents the lookback distance, the actual distance light has traveled. The evolution of the universe is described by the Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}, \quad (21)$$

where ρ is the energy density of the universe.

(e) For a matter-dominated universe with pressure $p = 0$, use the energy conservation equation

$$\dot{\rho} + 3\frac{\dot{a}}{a}\rho = 0 \quad (22)$$

to derive $\rho(a)$.

Rewriting this energy conservation, you get

$$\frac{d\rho}{\rho} = -3\frac{da}{a}. \quad (23)$$

Integrating, we obtain

$$\begin{aligned} \rho &\propto a^{-3}(t) \\ \Rightarrow \rho(t) &= \rho_0 a^{-3}(t). \end{aligned} \quad (24)$$

This means the energy density of a matter-dominated universe decreases as the universe expands.

(f) Substitute this into the Friedmann equation and solve for $a(t)$, particularly for a flat universe ($k = 0$).

Substituting $\rho(t) = \rho_0 a^{-3}(t)$ into the Friedmann equation for $k = 0$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho_0 a^{-3}(t). \quad (25)$$

Defining $H_0^2 = 8\pi G\rho_0/3$, we get

$$\dot{a}(t) = H_0 a^{-1/2}(t). \quad (26)$$

Solving this, we get

$$\begin{aligned} a(t) &\propto t^{2/3} \\ a(t) &= \left(\frac{t}{t_0}\right)^{2/3} \quad (\because a_0 = 1) \end{aligned} \quad (27)$$

This is the time evolution of the scale factor in a matter-dominated universe.